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Chapter IX

~~Involutions~~
Algebras Without Finiteness Conditions

§1 Alternative division algebras

This ~~Appendix~~ Chapter is devoted to the study of alternative algebras without any finiteness restrictions. We begin in this section by completely determining all alternative division algebras; they turn out to be either associative or Cayley algebras, just as before. Of course, the associative division algebras are not completely classified, but from the standpoint of alternative algebras we consider our task finished if we have reduced our problem to one about associative algebras.

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We recall the Nucleus = Center Theorem §II.1.10: if A is an alternative division algebra, then either A is associative or its nucleus and center coincide, $N(A) = C(A)$. It might happen that the nucleus and center reduce to zero; this is made unlikely by the following striking result, which provides a supply of nuclear elements.

1.1 (Fourth Power Theorem) In an alternative algebra the fourth power of any commutator lies in the nucleus,

$$[x,y]^4 \in N .$$

When $[x,y]$ is not a zero divisor, already the second power lies in the nucleus, $[x,y]^2 \in N$.

Proof. Set $z = [x,y]$; we will show

$$z[z^2,a,b] = [z^2,a,b]z = 0 \qquad (1.2)$$

for all a,b. If z is not a zero divisor this implies all

$[z^2,a,b] = 0$ and $z^2 \in N$, while in general by Middle Bumping it implies $[z^4,a,b] = z^2[z^2,a,b] + [z^2,a,b]z^2 = 0$.

¶ We begin by proving

$$\boxed{(*) \qquad [x,y,z\circ a] = z\circ[x,y,a] = 0.}$$

By I.3.7-8 both $D_x = [x,\cdot]$ and $A_{x,y} = [x,y,\cdot]$ are derivations of Jordan products. Here $A_{x,y}$ kills $z = [x,y]$ by Artin's Theorem, which establishes the first equality. Also

$$\begin{aligned}
[x,y,[x,y]\circ a] &= [x,y,[x,y\circ a] - y\circ[x,a]] \\
&= \{-[x,y\circ a,[x,y]]\} - \{[x,y,y]\circ[x,a] + y\circ[x,y,[x,a]]\} \\
&= -\{[x,y,[x,y]]\circ a + y\circ[x,a,[x,y]]\} - \{0 - y\circ[x,a,[x,y]]\} \\
&= 0
\end{aligned}$$

Since D_x, $A_{x,y}$, $A_{x,[x,y]}$ are Jordan derivations and since linearization of $[x,y,[x,y]] = 0$ yields $[x,y,[x,c]] = -[x,c,[x,y]]$.

¶ From (*) $[z^2,a,[x,y,b]] = z\circ[z,a,[x,y,b]] = [z,a,z\circ[x,y,b]] - [z,a,z]\circ[x,y,b] = 0$,

$$\boxed{(**) \qquad [z^2,a,[x,y,b]] = 0}$$

¶ Linearizing $[[x,y]^2,x,y] = 0$ yields $[[x,y]^2,x,a] = -[[x,y]\circ[x,a],x,y]$ which vanishes by (*), so $[z^2,x,a] = 0$, and finally $[z^2,y,a] = 0$. From these

$$\begin{aligned}
&(***) \qquad [z^2,a,xa] = a[z^2,a,x] = 0 \\
&(****) \qquad [z^2,a,(xy)a] = a[z^2,a,xy] = -a[z^2,y,xa] = 0
\end{aligned}$$

(using linearized (***)). ¶ We can use the linearizations of these relations to flip factors from one term to another inside an associator:

$$\begin{aligned}
[z^2,a,zb] &= [z^2,a,(xy)b] - [z^2,a,(yx)b] \\
&= [z^2,a,x(yb)] + [z^2,b,(yx)a] \qquad \text{(by (**), (****)}
\end{aligned}$$

$$= -[z^2, (yb), xa] + [z^2, b, y(xa)] \qquad \text{(by (**), (**))}$$
$$= +[z^2, xa, yb] - [z^2, (xa), yb] \qquad \text{(by (***))}$$
$$= 0.$$

Then $0 = [z^2, a, zb] = z \circ [z, a, zb] = z \circ [z, a, b] z = \{z \circ [z, a, b]\} z = [z^2, a, b] z$ by middle and left bumping. Dually $z[z^2, a, b] = 0$, and (1.2) is established ∎

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Thus supplied with central elements, we can establish the structure of an arbitrary alternative division algebra.

1.3 (Bruck-Kleinfeld-Skornyakov Theorem) An alternative division algebra is either associative or a Cayley algebra over its center.

Proof. Assume throughout that the alternative division algebra A is <u>not</u> associative. We wish to show A is a degree 2 algebra over its center. But for any element x we can actually write down a quadratic equation it satisfies:

$$(1.4) \qquad \alpha x^2 - \beta x + \gamma 1 = 0 \qquad \text{(Hall's identity)}$$

where
$$\begin{cases} \alpha = [x,y]^2 \\ \beta = [x,y]^2 x - [x,y]x[x,y] = [x,y] \circ [x,y'] \\ \gamma = [x,y]x[x,y]x = [x,y']^2 \end{cases}$$

(here $y' = yx$ has $[x,y'] = xyx - yxx = [x,y]x$). By Artin's theorem the above equation holds identically in x and y. We must show the coefficients lie in the center and can be chosen nontrivial.

Since a division algebra has no zero divisors, $\alpha = [x,y]^2$

and $\gamma = [x,y']^2$ lie in the nucleus N by the Fourth Power Theorem, and so does the linearization $\beta = [x,y]\circ[x,y']$.

By the Nucleus = Center Theorem and the assumed nonassociativity of $\langle A\rangle, \alpha, \beta, \gamma$ lie in $N = C$.

Thus every element x satisfies a quadratic equation over C. The only trouble is that it might be a trivial equation. It will be nontrivial if $[x,y] \neq 0$ (since then $\alpha = [x,y]^2 \neq 0$), so by proper choice of y we can get a nontrivial equation unless $[x,y] = 0$ for all possible y. But such an x already lies in C by the Commutativity-Implies-Centrality ~~Lemma~~ Theorem III.4.1 (since A has no nilpotents), in which case $x = \delta$ satisfies $x^2 - 2\delta x + \delta^2 = 0$.

So far we know every x satisfies an equation

$$(1.4) \qquad x^2 - t(x)x + n(x)1 = 0$$

for some coefficients $t(x), n(x)$ in the center C. If we can show t is linear then A will be degree 2 over C; since A is already semiprime (being a division algebra), by the Equivalence and Hurwitz Theorems VII.2.14 and VII.4.1, A will have to be a Cayley algebra over C.

But in VII.1.6 we saw that (1.4) and the fact that A has no zero divisors force it to be degree 2. ∎

1.5 Remark For a different proof of the last statement, recall that we have seen t is automatically linear if $|C| > 2$. But if $C = \mathbb{Z}_2$ is finite then any two elements $x,y \in A$ generate a finite associative subalgebra $C[x,y] = C1 + Cx + Cy + Cxy$ $= C1 + Cx + Cy + Cyx$ (note $xy + yx = (x+y)^2 - x^2 - y^2$ $\in C(x+y) + Cx + Cy + C1$) without zero divisors, which must be a finite field by Wedderburn's Theorem. In particular, x and y commute. This holds for all x and y, so A is commutative, hence associative by III.4.1, contrary to our hypothesis. ∎

The preceding remark once again shows

1.6 (Wedderburn Theorem) Any finite alternative division ring is a finite (commutative, associative) field. ∎

Exercises IX.1

Give an alternate proof of the 4th Power Theorem as follows.

1.1 Show (*) by writing —— no space —— $z \circ a = xy \circ a$

$= (ay)x - y(xa) = \{L_{xy} - L_y L_x\}a + \{R_{xy} - R_x R_y\}a$ and $\Delta_{xy} =$
$L_{xy} - L_x L_y = -R_{xy} + R_y R_x$, then using left and right Moufang
to prove $\Delta_{xy}(z \circ a) = 0$.

1.2 Deduce $[z^2, x, a] - [z^2, y, a] = 0$, and linearize to obtain
$x[z^2, a, b] = [z^2, a, bx]$, $y[z^2, a, b] = [z^2, ay, b]$.

1.3 Use the HIDING TRICK to show $x\{y[z^2, a, b]\} = y\{x[z^2, a, b]\}$.

1.4 Use $(xy)[z^2, a, b] = [x, y, [z^2, a, b]] + x\{y[z^2, a, b]\}$ to show
$z[z^2, a, b] = 0$.

Still another proof was given in Problem Set III 2.1.

IX.1.1 ~~XAI.1~~ Problem Set on Domains

Go back through the proof of the Bruck-Kleinfeld-Skornyakov Theorem, making whatever additional arguments are necessary, to establish

(Theorem) If A is an alternative algebra without zero divisors, then A is either associative or an order in a Cayley algebra.

Suggested steps are:

1. Show that if A is not associative it is of degree 2 over $C = N$, and semiprime.
2. Show C is an integral domain with quotient field $\tilde{C}$, and that A can be imbedded as a C-order in its central closure $\tilde{A} = A \otimes_C \tilde{C}$. Show $\tilde{A}$ has no zero divisors, and is of degree 2 over $\tilde{C}$ iff is degree 2 over C.
3. If $\tilde{A}$ is not associative but is a domain of degree 2 over a field $\tilde{C}$, show $\tilde{A}$ is a Cayley algebra, so our original A is an order in a Cayley algebra.

IX.1.2 #AII#2 Problem Set on the Fourth Power Theorem

1. Prove that $[x,m][y,z,w][x,n] = [x,m][x,n][y,z,w] = 0$ for all $x,y,z,w \in A$ and $n,m \in N(A)$.

2. Prove any associator $a = [x,y,z]$ satisfies $a^2 = [x,y,az] - c(az) = -B_{x,y}(az)$ for $c = [x,y]$, $B_{x,y} = L_{xy} - L_y L_x$.

3. Prove that $[[x,y]^2,z,w]^2 = 0$ holds in all alternative algebras.

4. Conclude that if A has no nilpotent elements, the square $[x,y]^2$ of any commutator lies in the nucleus.

This improves on the Fourth Power Theorem (instead of no zero divisors we need only no nilpotents). However, we cannot generalize the Nucleus = Center Theorem from the case of no zero divisors to the case of no nilpotents: if D is a central associative division algebra and C a Cayley division algebra then $A = D \boxplus C$ has no nilpotents but has nucleus $N(A) = D \oplus \Phi$ different from A or $C(A) = \Phi \oplus \Phi$.